Polynomial Optimization

Sums of squares and moments

Part 2

Monique Laurent

POEMA 1st Workshop — Florence, January 2020

Solving polynomial equations

Finite rank moment matrices

Recap on polynomial optimization

Given polynomials $p, g_1, \cdots, g_m$, compute:

$$p_{\min} = \inf_{x \in K} p(x), \qquad K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}.$$

Recall:

- $\overline{p} = (p_\alpha)_\alpha$ is the vector of coefficients of p.
- $[x]_\infty = (x^\alpha)_{\alpha \in \mathbb{N}^n}$ is the vector of monomials.
- $p(x) = \sum_\alpha p_\alpha x^\alpha = \overline{p}^\mathsf{T}[x]_\infty$.
- Define the set $\mathcal{C}_\infty(K) = \mathrm{conv}\{[x]_\infty : x \in K\}$.
- Then, $p_{\min} = \inf_{x \in K} \overline{p}^\mathsf{T}[x]_\infty = \inf_{y=(y_\alpha) \in \mathbb{R}^{\mathbb{N}^n}} \{\overline{p}^\mathsf{T} y : y \in \mathcal{C}_\infty(K)\}$.

Goal: Describe the set $\mathcal{C}_\infty(K)$

Plan of the lecture

1. For an ideal $I \subseteq \mathbb{R}[x]$, basic facts about the quotient algebra $\mathcal{A} = \mathbb{R}[x]/I$:
   - Relate $\dim \mathcal{A}$ and $|V_{\mathbb{C}}(I)|$.
   - The eigenvalue method to find $V_{\mathbb{C}}(I)$.

2. Moment matrices $M(y)$:
   - The kernel of $M(y)$ is an ideal I.
   - If $M(y) \succeq 0$ then I is a real radical ideal.

3. Characterization of the sequences $y \in \mathcal{C}_\infty(K)$ in terms of positivity and finite rank condition on the moment matrix $M(y)$.

Ideals and varieties

Let $I \subseteq \mathbb{C}[x]$ be an ideal.

- $V_{\mathbb{C}}(I) = \{x \in \mathbb{C}^n : f(x) = 0 \ \forall f \in I\}$ is the **complex variety** of I.
- $V_{\mathbb{R}}(I) = V_{\mathbb{C}}(I) \cap \mathbb{R}^n$ is the **real variety** of I.
- If $I = (h_1, \cdots, h_m)$ is generated by $h_1, \cdots, h_m$, the elements $v \in V_{\mathbb{C}}(I)$ are the **common complex roots** of $h_1, \cdots, h_m$.
- The **vanishing ideal** of a subset $V \subseteq \mathbb{C}^n$ is

$$\mathcal{I}(V) = \{f \in \mathbb{C}[x] : f(v) = 0 \ \forall v \in V\}.$$

- Clearly: $I \subseteq \mathcal{I}(V_{\mathbb{C}}(I))$.

Radical ideals and Hilbert's Nullstellensatz

- The **radical** of I is the ideal:

$$\sqrt{I} = \{f \in \mathbb{C}[x] : f^m \in I \text{ for some } m \in \mathbb{N}\}.$$

- Clearly: $I \subseteq \sqrt{I} \subseteq \mathcal{I}(V_{\mathbb{C}}(I))$.

- **Example:** For $I = (x^2)$, $V_{\mathbb{C}}(I) = \{0\}$, $x \in \sqrt{I}$, but $x \notin I$.

- **Hilbert's Nullstellensatz:** $\sqrt{I} = \mathcal{I}(V_{\mathbb{C}}(I))$.

  A polynomial f vanishes at all common complex roots of I if and only if some power of f belongs to I.

- **Definition:** The ideal I is said to be **radical** if $I = \sqrt{I}$ or, equivalently, if $I = \mathcal{I}(V_{\mathbb{C}}(I))$.

Real radical ideals & the Real Nullstellensatz

Let I be an ideal in $\mathbb{R}[x]$.

- The **real radical** of I is the ideal:

$$\sqrt[\mathbb{R}]{I} = \{f \in \mathbb{R}[x] : f^{2m} + s \in I \quad \text{for some } m \in \mathbb{N}, s \in \Sigma\}.$$

- Clearly: $I \subseteq \sqrt[\mathbb{R}]{I} \subseteq \mathcal{I}(V_{\mathbb{R}}(I))$.
- **Example:** For $I = (x^2 + y^2)$, we have $V_{\mathbb{R}}(I) = \{(0,0)\}$. Then, $x, y \in \sqrt[\mathbb{R}]{I}$, but $x, y \notin I$.
- **Real Nullstellensatz:** $\sqrt[\mathbb{R}]{I} = \mathcal{I}(V_{\mathbb{R}}(I))$.
- **Definition:** The ideal I is said to be **real radical** if $I = \sqrt[\mathbb{R}]{I}$ or, equivalently, if $I = \mathcal{I}(V_{\mathbb{R}}(I))$.

Some technical lemmas

1. **Lemma 1:** I is **real radical** if and only if

$$\forall f_1, \cdots, f_m \in \mathbb{R}[x] \quad f_1^2 + \cdots + f_m^2 \in I \Longrightarrow f_1, \cdots, f_m \in I.$$

2. **Lemma 2:** If $I \subseteq \mathbb{R}[x]$ is a real radical ideal and $|V_{\mathbb{R}}(I)| < \infty$, then

$$V_{\mathbb{C}}(I) = V_{\mathbb{R}}(I).$$

3. **Lemma 3:** Let $V \subseteq \mathbb{C}^n$ be a finite set. There exist **interpolation polynomials** $p_v \in \mathbb{C}[x]$ for $v \in V$, i.e., satisfying:

$$p_v(u) = \delta_{u,v} \quad \forall u, v \in V.$$

   If $V = \overline{V}$, the interpolation polynomials p_v can be chosen in $\mathbb{R}[x]$.

   For any polynomial $f \in \mathbb{C}[x]$, $\quad f - \sum_{v \in V_{\mathbb{C}}(I)} f(v) p_v \in \mathcal{I}(V)$.

The quotient algebra $\mathbb{R}[x]/I$

The quotient algebra $\mathbb{C}[x]/I$

Let $I \subseteq \mathbb{C}[x]$ be an ideal.

The quotient $\mathcal{A} = \mathbb{C}[x]/I$ is an algebra:

- Elements: cosets $[f] = f + I = \{f + q : q \in I\}$.
- Addition: $[f] + [g] = [f + g]$.
- Scalar multiplication: $\lambda[f] = [\lambda f]$.
- Multiplication: $[f][g] = [fg]$.

A set $\mathcal{B} = \{[b_1], [b_2], \ldots\} \subseteq \mathcal{A}$ is a **linear basis** of $\mathcal{A}$ if any polynomial $f \in \mathbb{C}[x]$ can be written (uniquely) as

$$[f] = \sum_i \lambda_i [b_i] \qquad \text{i.e.,} \qquad f = \sum_i \lambda_i b_i + q,$$

where $\lambda_i \in \mathbb{C}$ and $q \in I$.

The dimension of $\mathcal{A} = \mathbb{C}[x]/I$

Lemma 4: Let p_v ($v \in V_{\mathbb{C}}(I)$) be interpolation polynomials at $V_{\mathbb{C}}(I)$ and define

$$\mathcal{L} := \{[p_v] : v \in V_{\mathbb{C}}(I)\}.$$

- $\mathcal{L}$ is linearly independent in $\mathcal{A}$.
- $\mathcal{L}$ is generating in $\mathbb{C}[x]/\mathcal{I}(V_{\mathbb{C}}(I))$.
- $\mathcal{L}$ is a basis of $\mathcal{A}$ if I is radical.

Theorem:

1. $\dim \mathcal{A} < \infty \iff |V_{\mathbb{C}}(I)| < \infty$.
2. Assume $|V_{\mathbb{C}}(I) < \infty$. Then,

$$|V_{\mathbb{C}}(I)| \leq \dim \mathcal{A},$$

with equality if and only if the ideal I is radical (i.e., $I = \sqrt{I}$).

Multiplication operators and roots of equations

Given a polynomial h, define the **'multiplication by h' linear map:**

$$\begin{array}{rccc} m_h: & \mathcal{A} & \to & \mathcal{A} \\ & [f] & \mapsto & [fh]. \end{array}$$

Theorem: Assume $|V_{\mathbb{C}}(I)| < \infty$.

1. Let $\mathcal{B} = \{[b_1], \cdots, [b_N]\}$ be a basis of $\mathcal{A}$ and let M_h be the matrix of m_h in the basis $\mathcal{B}$. For $v \in V_{\mathbb{C}}(I)$, the vector $[v]_{\mathcal{B}} = (b_i(v))_{i=1}^N$ is a left **eigenvector** of M_h:

$$[v]_{\mathcal{B}}{}^{\mathsf{T}} M_h = h(v)[v]_{\mathcal{B}}{}^{\mathsf{T}}.$$

2. $\{h(v) : v \in V_{\mathbb{C}}(I)\}$ is the set of all the **eigenvalues** of M_h.

$\rightsquigarrow$ **Can compute $V_{\mathbb{C}}(I)$ via the eigenvalues/eigenvectors of M_h**
for random linear $h = \sum_{i=1}^n h_i x_i$ $\rightsquigarrow$ the eigenvalues $h(v)$ are all distinct, so one can recover $[v]_{\mathcal{B}}$ and thus v.

Univariate example

- Let $I = (x^3 - 6x^2 + 11x - 6)$ be generated by
$$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3).$$
- Thus: $V_{\mathbb{C}}(I) = \{1, 2, 3\}$.
- The set $\mathcal{B} = \{[1], [x], [x^2]\}$ is a basis of $\mathcal{A} = \mathbb{R}[x]/I$.
- 'Multiplication by x' matrix (*companion matrix*):
$$M_x = \begin{array}{c} \\ [1] \\ [x] \\ [x^2] \end{array} \begin{array}{c} \begin{array}{ccc} [x] & [x^2] & [x^3] \end{array} \\ \begin{pmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix} \end{array}$$
- M_x^{T} has three eigenvectors:
  1. $(1, 1, 1)$ with eigenvalue $\lambda = 1$,
  2. $(1, 2, 4)$ with eigenvalue $\lambda = 2$,
  3. $(1, 3, 9)$ with eigenvalue $\lambda = 3$.

Eigenvectors are indeed of the form $[v]_{\mathcal{B}} = (1, v, v^2)$ for $v \in \{1, 2, 3\}$.

Finite rank moment matrices

Recap on moment matrices

Definition: Let $y = (y_\alpha)_\alpha$ be a sequence of real numbers indexed by $\mathbb{N}^n$.

1. Define the corresponding **linear functional** L_y on $\mathbb{R}[x]$:

$$
\begin{array}{llcl}
L_y : & \mathbb{R}[x] & \to & \mathbb{R} \\
 & x^\alpha & \mapsto & L_y(x^\alpha) = y_\alpha \\
 & f = \sum_\alpha f_\alpha x^\alpha & \mapsto & L_y(f) = \sum_\alpha f_\alpha y_\alpha .
\end{array}
$$

2. Define the **moment matrix** $M(y)$, as the real symmetric matrix indexed by $\mathbb{N}^n$ with

$$M(y)_{\alpha,\beta} = L_y(x^\alpha x^\beta) = y_{\alpha+\beta} \quad \forall \alpha, \beta \in \mathbb{N}^n.$$

Example: If $y = [v]_\infty$ with $v \in \mathbb{R}^n$, then L_y is the **evaluation at** v:

$$L_y(f) = f(v) \qquad \forall f \in \mathbb{R}[x],$$

and

$$M(y) = (v^{\alpha+\beta})_{\alpha,\beta}.$$

Positivity conditions for $M(y)$ and L_y

Lemma 5: Let $y = (y_\alpha)_\alpha$ and L_y the associated linear functional on $\mathbb{R}[x]$.

For $f, g \in \mathbb{R}[x]$:

$$L_y(f^2) = \overline{f}^{\mathsf{T}} M(y) \overline{f}, \quad L_y(gf^2) = \overline{f}^{\mathsf{T}} M(gy) \overline{f},$$

where $gy \in \mathbb{R}^{\mathbb{N}^n}$ is the new sequence with α-th entry

$$(gy)_\alpha = L_y(gx^\alpha) = L_y\Big(\Big(\sum_\gamma g_\gamma x^\gamma\Big) x^\alpha\Big) = \sum_\gamma g_\gamma y_{\alpha+\gamma} \quad \forall \alpha \in \mathbb{N}^n.$$

Therefore,

$$L_y \geq 0 \text{ on } \Sigma \iff M(y) \succeq 0,$$

$$L_y \geq 0 \text{ on } g\Sigma \iff M(gy) \succeq 0.$$

The kernel of $M(y)$ is an ideal

Lemma 6: Let $y = (y_\alpha)_\alpha$ and L_y the associated linear functional on $\mathbb{R}[x]$. Set

$$I := \{f \in \mathbb{R}[x] : L_y(hf) = 0 \ \forall h \in \mathbb{R}[x]\}.$$

Then:

1. A polynomial f belongs to I if and only if its coefficient vector $\overline{f}$ belongs to the kernel of $M(y)$.

   So we can write: $I = \ker M(y)$.

2. I is an **ideal** in $\mathbb{R}[x]$.

3. If $M(y) \succeq 0$ then I is a **real radical ideal**.

   using the facts:

   - $f \in I \iff L(f^2) = 0$.
   - I is real radical if and only if $\sum_i f_i^2 \in I \implies f_i \in I$ for all i

Characterization of the set $\mathcal{C}_\infty(K)$

Finite rank moment matrix theorem: [Curto-Fialkow 1996, 2000]

Let $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$.

Let $y = (y_\alpha)_\alpha$ and L_y the corresponding linear functional on $\mathbb{R}[x]$.

The following assertions are equivalent:

(1) $y \in \mathcal{C}_\infty(K)$, i.e., $y = \sum_{i=1}^r \lambda_i [v_i]_\infty$ for $\lambda_i > 0$, $\sum_i \lambda_i = 1$, $v_i \in K$,

i.e., y has a representing measure which is **finite atomic and supported by** K.

(2) $y_0 = 1$, $\text{rank}\, M(y) < \infty$, $M(y) \succeq 0$ and $M(g_j y) \succeq 0$ for $j \in [m]$.

(3) $y_0 = 1$, $\text{rank}\, M(y) < \infty$ and $L_y \geq 0$ on $\Sigma + g_1\Sigma + \cdots + g_m\Sigma$.

Proof of the implication (2) $\Longrightarrow$ (1)

1. $I := \ker M(y)$ is a real radical ideal. [since $M(y) \succeq 0$]
2. $\dim \mathbb{R}[x]/I = \mathrm{rank}\ M(y) =: r$.
3. $V_{\mathbb{C}}(I) = \{v_1, \cdots, v_r\} \subseteq \mathbb{R}^n$. [$I$ real radical with finite variety]
4. Let $p_{v_1}, \cdots, p_{v_r} \in \mathbb{R}[x]$ be interpolation polynomials at $v_1, \ldots, v_r$. Their cosets form a basis of $\mathbb{R}[x]/I$.
5. $L_y = \sum_{i=1}^r L_y(p_{v_i})\mathcal{L}_{v_i}$. [$\mathcal{L}_{v_i}$ = 'evaluation at v_i' liner functional]
6. $L_y(p_{v_i}) > 0$. [since $p_{v_i} - (p_{v_i})^2 \in I$]
7. $\sum_{i=1}^r L_y(p_{v_i}) = 1$. [since $1 - \sum_{i=1}^r p_{v_i} \in I$]
8. $v_1, \cdots, v_r \in K$. [since $0 \leq L_y(g_j(p_{v_i})^2) = L_y(p_{v_i})g_j(v_i)$]

   We are done:
$$y = \sum_{i=1}^{r} L_y(p_{v_i})[v_i]_\infty \in \mathcal{C}_\infty(K).$$

Polynomial optimization:
Stopping criterion (flatness condition)
Extracting global minimizers

Flat extension of matrices

Lemma 1: Consider a matrix in block form: $X = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}$.
Assume $\mathrm{rank}\, X = \mathrm{rank}\, A$ (call X a **flat extension** of A). Then

$$\ker X = \ker(A\ B)$$

Lemma 2: Given $y \in \mathbb{R}^{\mathbb{N}^n_{2s}}$, assume $\mathrm{rank}\, M_s(y) = \mathrm{rank}\, M_{s-1}(y)$,
i.e., $M_s(y)$ is a **flat extension of** $M_{s-1}(y)$ (then call y **flat**).
Then, for any polynomials f, g:

$$f \in \ker M_s(y),\ \deg(fg) \leq s \implies fg \in \ker M_s(y).$$

Proof:

- Suffices to show the result for $g = x_i$ (then iterate).
- Suffices to show $L(u(fx_i)) = 0 \ \ \forall u \in \mathbb{R}[x]_{s-1}$. [by Lemma 1]

Indeed: $L(u(fx_i)) = L((x_i u)f) = 0$ as $\deg(x_i u) \leq s$ and $f \in \ker M_s(y)$.

Hence: $\ker M_s(y)$ behaves like a **'truncated' ideal**.

Flat extension of moment matrices

Theorem 1: [Flat extension theorem of Curto-Fialkow 1996]
Given $y \in \mathbb{R}^{\mathbb{N}^n_{2s}}$, assume:

$$\mathrm{rank} M_s(y) = \mathrm{rank} M_{s-1}(y).$$

Then one can **extend** y **to a sequence** $\tilde{y} \in \mathbb{R}^{\mathbb{N}^n}$ satisfying:

$$\mathrm{rank}\ M(\tilde{y}) = \mathrm{rank}\ M_s(y).$$

Moreover, the ideal $I = \ker M(\tilde{y})$ satisfies:

(1) If $\{\alpha_1, \cdots, \alpha_r\} \subseteq \mathbb{N}^n_{s-1}$ indexes a column base of $M_{s-1}(y)$, then $\{[x^{\alpha_1}], \cdots, [x^{\alpha_r}]\}$ is a base of $\mathbb{R}[x]/I$.

(2) I is generated by the polynomials in $\ker M_s(y)$:

$$I = (\ker M_s(y)).$$

Sketch of proof of Theorem 1

Goal: Find a matrix $M = \begin{pmatrix} M_s(y) & B \\ B^T & C \end{pmatrix}$, indexed by $\mathbb{N}^n_{s+1}$, satisfying:

(1) M is a **flat extension** of $M_s(y)$: $\mathrm{rank}\ M = \mathrm{rank}\ M_s(y) =: r$

(2) M is a **moment matrix**: $M_{\alpha,\beta} = M_{\alpha',\beta'}$ for all $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^n_{s+1}$

Key ideas:

- Let $\mathcal{B} = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\} \subseteq \mathbb{R}[x]_{s-1}$ index a maximum linearly independent set of columns of $M_s(y)$.

- Let $|\gamma| = t + 1$. Say, $\gamma_i \geq 1$. **If such M exists it must satisfy:**
  $x^{\gamma - e_i} - r(x) \in \ker M_s(y) \subseteq \ker M$, for some $r \in \mathrm{Span}(\mathcal{B})$
  and thus $x_i^\gamma - x_i r(x) = x_i(x^{\gamma - e_i} - r(x)) \in \ker M$. [by Lemma 2]
  This permits to define the γth column of M in terms of columns of $M_s(y)$ (and thus to define B and C).

- Remains to verify that this is a **good** definition:
  - it does not depend on the choice of index i such that $\gamma_i \geq 1$,
  - and the matrix M obtained in this way is a moment matrix.

Stopping criterion and extracting global minimizers

- K_p^*: set of all **global minimizers** of p in K.
- $d_p = \lceil \deg(p)/2 \rceil$, $d_K = \max\{\lceil \deg(g_j)/2 \rceil : j \in [m]\}$.

Theorem 2: [Lasserre 2001]
Let L be an optimal solution to

$$p_{\mathrm{mom},t} = \inf\{L(p) : L \in (\mathbb{R}[x]_{2t})^*, L(1) = 1, L \geq 0 \text{ on } \mathcal{M}(\mathbf{g})_{2t}\}$$

with associated sequence $y = (L(x^\alpha))_{\alpha \in \mathbb{N}^n_{2s}}$. Assume:

$\mathrm{rank}\ M_s(y) = \mathrm{rank}\ M_{s-d_K}(y)$ for some s with $\max\{d_p, d_K\} \leq s \leq t$.

Then:

(1) The relaxation is exact: $p_{\mathrm{mom},t} = p_{\min}$.

(2) All common roots of the polynomials in $\ker M_s(y)$ are real and they are global minimizers: $V_{\mathbb{C}}(\ker M_s(y)) \subseteq K_p^*$.

(3) If L is an optimal solution for which the matrix $M_t(y)$ has maximum possible rank, then:

$$V_{\mathbb{C}}(\ker M_s(y)) = K_p^*.$$

Proof of Theorem 2

1. **Apply the 'flat extension theorem':** There exists a sequence $\tilde{y} \in \mathbb{R}^{\mathbb{N}^n}$ **extending the subsequence** $(y_\alpha)_{|\alpha| \leq 2s}$ satisfying:
   (a) $\text{rank } M(\tilde{y}) = \text{rank } M_s(y) = \text{rank } M_{s-d_K}(y) =: r$.
   (b) $I := \ker M(\tilde{y}) = (\ker M_s(y))$.
   (c) If $\{\alpha_1, \cdots, \alpha_r\} \subseteq \mathbb{N}^n_{s-d_K}$ indexes a column basis of $M_{s-d_K}(y)$, then $\mathcal{B} = \{[x^{\alpha_1}], \cdots, [x^{\alpha_r}]\}$ is a basis of $\mathbb{R}[x]/I$.
2. **Apply the 'finite rank moment matrix theorem':**
   (a) $V_{\mathbb{C}}(I) = V_{\mathbb{C}}(\ker M_s(y))) = \{v_1, \cdots, v_r\} \subseteq \mathbb{R}^n$.
   (b) $\tilde{y} = \sum_{i=1}^r \lambda_i [v_i]_\infty$, where $v_i \in K$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$.
   (c) $(y_\alpha)_{|\alpha| \leq 2s} = \sum_{i=1}^r \lambda_i [v_i]_{2s}$ where $v_i \in K$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$.
3. **Hence:** $p_{\mathrm{mom},t} = p_{\min}$ and $v_1, \cdots, v_r$ are global minimizers.

Proof of Theorem 2 (continued)

Assume now L_y is an optimal solution for which $\mathrm{rank}\ M_t(y)$ is **maximum**.

That is, for any other optimal solution z to $p_{\mathrm{mom},t}$, we have $\mathrm{rank}\ M_t(z) \leq \mathrm{rank}\ M_t(y)$.

This implies
$\ker M_t(y) \subseteq \ker M_t(z)$, and thus
$\ker M_s(y) \subseteq \ker M_s(z)$,
$V_{\mathbb{C}}(\ker M_s(z)) \subseteq V_{\mathbb{C}}(\ker M_s(y)) = V_{\mathbb{C}}(I) = \{v_1, \ldots, v_r\}$. .

Let $x^* \in K$ be a global minimizer of p.
Then $z = [x^*]_{2t}$ is an optimal solution of $p_{\mathrm{mom},t}$.
Therefore, $\{x^*\} = V_{\mathbb{C}}(\ker M_s(z)) \subseteq \{v_1, \ldots, v_r\}$.

Some observations

- **Use the 'eigenvalue method' to extract the global minimizers:** via the eigenvectors of the multiplication matrix M_h for $h = \sum_{i=1}^n h_i x_i$.

All needed information is contained in $M_s(y)$: the entries of $M_h = \sum_{i=1}^n h_i M_{x_i}$ can be derived directly by expressing each $[x_i x^{\alpha_j}]$ in the basis $\mathcal{B} = \{[x^{\alpha_1}], \ldots, [x^{\alpha_r}]\}$.

- If the flatness condition holds then p has **finitely many global minimizers** in K.

The converse is **not** true!

Example: Let $K = \{x : \sum_{i=1}^n x_i^2 \leq 1\}$ and assume p is homogeneous, $p > 0$ on $\mathbb{R}^n \setminus \{0\}$, p is not SOS. Then, $p_{\min} = 0$, attained **only at** 0. But, $\quad p_{\text{sos},t} = p_{\text{mom},t} < p_{\min} = 0 \quad$ for all $t \geq d_p$.

Fact: $p \in \mathcal{M}(1 - \sum_i x_i^2) \implies p \in \Sigma$

- The flatness condition holds **generically**. [Nie 2014]