EXERCISES POEMA LEARNING WEEK 2

TIMO DE WOLFF

Exercise. *Let*

$$f(\boldsymbol{x}) \;=\; f_{\boldsymbol{\beta}}\boldsymbol{x}^{\boldsymbol{\beta}} + \sum_{j=0}^{n} f_{\boldsymbol{\alpha(j)}}\boldsymbol{x}^{\boldsymbol{\alpha(j)}}$$

be a (nonnegative) circuit polynomial as defined in the lecture. Then $\boldsymbol{s} \in \mathbb{R}^n_{>0}$ *is a unique minimizer of* f *if and only if for all* $j \in [n]$ *it holds that:*

$$\boldsymbol{s}^{\boldsymbol{\alpha(j)}} \;=\; \frac{\lambda_j}{f_{\boldsymbol{\alpha(j)}}}.$$

Is $\boldsymbol{s}$ *also a root of* f*?*

Exercise. *A Hurwitz form is given by*

$$h(x_1,\ldots,x_{2n}) \;=\; \sum_{j=1}^{2n} x_j^{2n} - 2n \cdot \prod_{j=1}^{2n} x_j.$$

Show that every Hurwitz form is both a nonnegative circuit polynomial and a sum of squares.

Note: Indeed, this result can be seen as an "initial result on SONC" (or AM-GM / SAGE – choose your favorite name), proven by Hurwitz in 1891.

Exercise. *Show that the SONC cone* $C_{n,2d}$ *indeed is a convex, full-dimensional, pointed, closed cone in* $\mathbb{R}[x_1,\ldots,x_n]_{2d}$*.*

Note: Everything except the dimension of $C_{n,2d}$ *is straight forward to show. In order to show the statement about the dimension proceed as follows:*

(1) Consider a polynomial f *that is a sum of strictly positive circuit polynomials and that has full support (i.e., is in the interior of* $C_{n,2d}$*).*
(2) Take the 1*-norm of coefficients of polynomials to obtain a metric on* $\mathbb{R}[x_1,\ldots,x_n]_{2d}$ *(with a small argumentation).*
(3) Show that a full dimensional ball of polynomials in that metric around f *is contained in* $C_{n,2d}$*.*

Exercise. *Let* $A \subseteq \mathbb{N}^n$ *be a finite set and let* $(\mathbb{C}^*)^A$ *be the space of all* n*-variate complex (Laurent-)polynomials with support* A*. The A-discriminant is the algebraic variety in*

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$(\mathbb{C}^*)^A$ given by the Zariski-closure of all (Laurent-)polynomials for which there exists an $\boldsymbol{s} \in \mathbb{C}^n$ satisfying

$$f(\boldsymbol{s}) \;=\; \frac{\partial f}{\partial x_1}(\boldsymbol{s}) \;=\; \cdots \;=\; \frac{\partial f}{\partial x_n}(\boldsymbol{s}) \;=\; 0.$$

(1) Let A be a (simplicial) circuit (i.e., A is minimally affine dependent, $\mathrm{conv}(A)$ is a simplex) and let $f = \sum_{j=0}^{n} f_{\boldsymbol{\alpha(j)}} \boldsymbol{x}^{\boldsymbol{\alpha(j)}} - f_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$ be a nonnegative circuit polynomial supported on A with $|f_{\boldsymbol{\beta}}| = \Theta_f$. Show that f belongs ot the A-discriminant.

(2) Let now A be arbitrary and let $f = \sum_{i=1}^{k} f_i$ be a SONC supported on A such that every $f_i = \sum_{j=0}^{n} f_{\boldsymbol{\alpha(j)}_i} \boldsymbol{x}^{\boldsymbol{\alpha(j)_i}} - f_{\boldsymbol{\beta}_i} \boldsymbol{x}^{\boldsymbol{\beta_i}}$ is a nonnegative circuit polynomial with $|f_{\boldsymbol{\beta}_i}| = \Theta_{f_i}$. Is f still contained in the A-discriminant? If yes: Why? If no: Is there an additional assumption that can be made in order to ensure this containment?

Timo de Wolff, Technische Universität Braunschweig, Institut für Analysis und Algebra, AG Algebra, Universitätsplatz 2, 38106 Braunschweig, Germany

Email address: `t.de-wolff@tu-braunschweig.de`