Sum-of-Squares Hierarchy for Symmetric Formulations over Boolean hypercube

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ETH Zürich

POEMA 2nd workshop

INTRODUCTION

Let

$$\mathcal{G} := \{g_0 := 1, g_1, \ldots, g_m : \; g_i \in \mathbb{R}[x], \; i \in [m]\}$$

be the set of polynomials and

$$\mathcal{G}_+ = \{x \in \mathbb{R}^n \mid g(x) \geqslant 0 \;\; \forall_{g \in \mathcal{G}}\}$$

be the subset of $\mathbb{R}^n$ where polynomials in $\mathcal{G}$ are nonnegative.

Polynomial Optimization Problem

$$\mathbb{P}: \quad \min \; \{ \; f(x) \mid x \in \mathcal{G}_+ \; \}$$

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INTRODUCTION

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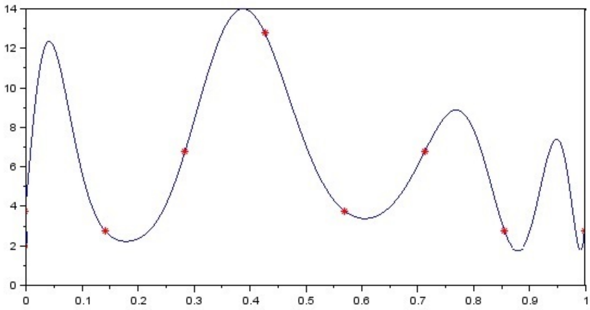

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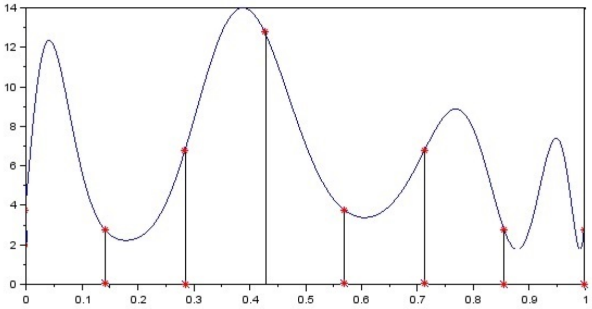

INTRODUCTION

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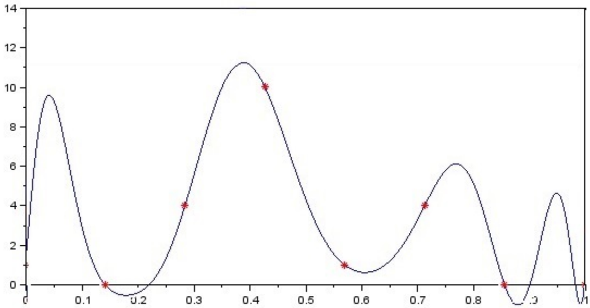
$$\mathbb{P}: \quad \min\{f(x) \mid x \in \mathcal{G}_+\} \qquad \Rightarrow \qquad \mathbb{P}: \quad \max\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{K}_{\mathcal{G}}\}$$



INTRODUCTION

$$\mathbb{P}: \quad \max\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{K}_{\mathcal{G}}\}$$

for $\mathcal{K}_{\mathcal{G}}$- the cone of polynomials that are nonnegative over the set $\mathcal{G}_+$

$$\mathbb{SOS}^{2d}: \quad \max\{\lambda \in \mathbb{R} \mid f - \lambda \in \Sigma_{\mathcal{G}}^{2d}\}$$

$$\Sigma_{\mathcal{G}}^{2d}: \{p \mid p = \underbrace{s_0}_{deg\ 2d+\lceil \text{deg}\ \mathcal{G}/2\rceil} + \sum_i \underbrace{g_i(x)s_i(x)}_{deg\ 2d+\lceil \text{deg}\ \mathcal{G}/2\rceil}\} \quad \text{and} \quad s_i \in SoS$$

$$s_i \in SoS_d \quad \Rightarrow \quad \text{SDP of size} \approx \binom{n+d}{d}.$$

$\mathbb{SOS}^{2d}$ - TWO OPTIONS

Primal

$$\max_{\lambda \in \mathbb{R}} \quad \lambda$$
$$s.t. \quad f - \lambda \in \Sigma^{2d}_{\mathcal{G}}$$

Dual

$$\min_{\tilde{\mathbb{E}}:\ \mathbb{R}[x]_{(2d+\deg \mathcal{G})} \to \mathbb{R}} \quad \tilde{\mathbb{E}}[f(x)]$$
$$s.t. \quad \tilde{\mathbb{E}}[1] = 1$$
$$\tilde{\mathbb{E}}[g_i h^2] \geqslant 0 \quad \forall\, i \in [m],\ h \in \mathbb{R}[x]$$
$$\deg(g_i h^2) \leqslant 2d + \lceil \deg \mathcal{G}/2 \rceil$$

$\mathbb{SOS}^{2d}$ - TWO OPTIONS

Primal

$$\max_{\lambda \in \mathbb{R}} \quad \lambda \qquad s.t. \quad f - \lambda \in \Sigma_{\mathcal{G}}^{2d}$$

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$$\min_{\tilde{\mathbb{E}}:\, \mathbb{R}[x]_{(2d+\deg \mathcal{G})} \to \mathbb{R}} \quad \tilde{\mathbb{E}}[f(x)]$$

$$s.t. \quad \tilde{\mathbb{E}}[1] = 1$$

$$\tilde{\mathbb{E}}[g_i h^2] \succeq 0 \quad \forall\, i \in [m],\ h \in \mathbb{R}[x]_{2d}$$

The Lasserre/SoS hierarchy

Consider a function $\mu: \{0,1\}^n \to \mathbb{R}$.
For $I \subseteq [n]$, $x_I \in \{0,1\}^n$, s.t. $(x_I)_i = 1$ iff $i \in I$,

$$\mu(x_I) = \tilde{\mathbb{E}}[\prod_{i \in I} x_i \prod_{[n] \setminus I} (1 - x_i)]$$

$\mu(x_I)$ can be thought of as an indicator variable of the solution I

SoS constraints: first type

$$\tilde{\mathbb{E}}[1] \;=\; \sum_{I \subseteq [n]} \mu(x_I) = 1.$$

$\mu(x_I)$ is often called a *pseudodistribution* as it mimics an actual distribution in some sense.

$$\tilde{\mathbb{E}} f = \sum_{I \in [n]} \mu(x_I) f(x_I)$$

THE LASSERRE/SOS HIERARCHY

To each $I \subseteq [n]$ we associate a vector $Z_I^t \in \mathbb{R}^{\binom{n}{\leqslant t}}$.

$$Z_I^t[J] = \begin{cases} 1, & \text{if } J \subseteq I, \quad |I|, |J| \leqslant t \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $Z_I^t(Z_I^t)^\top$ is PSD $\longrightarrow \mu(x_I) Z_I^t(Z_I^t)^\top$ is PSD when $\mu(x_I) \geqslant 0$.

SoS constraints: second type

$$\sum_{I \subseteq [n]} \mu(x_I) Z_I^t (Z_I^t)^\top \succeq 0.$$

THE LASSERRE/SOS HIERARCHY

Denote by $g_\ell(x_I)$ the constraint g_ℓ evaluated at the point $x \in \{0,1\}^n$ with $(x_I)_i = 1 \Leftrightarrow i \in I$.

For feasible points we have $g(x_I) \geqslant 0 \longrightarrow \mu(x_I) g(x_I) Z_I^t (Z_I^t)^\top$ should be PSD when $\mu(x_I) \geqslant 0$ since $Z_I^t (Z_I^t)^\top \succeq 0$.

SoS constraints: third type

$$\sum_{I \subseteq [n]} \mu(x_I) g(x_I) Z_I^t (Z_I^t)^\top \succeq 0.$$

INTRODUCTION

0/1 Integer program

$$\min_{x\in\{0,1\}^n} f(x)$$

$$g_i(x) \geqslant 0, \text{ for } i = [m]$$

$f,\ g_i$ - polynomials

Lasserre/SoS at level t

$$\mathbb{SOS}^{2t}: \min_{\mu:2^n\to\mathbb{R}} \sum_{I\subseteq[n]} f(x_I)\mu(x_I)$$

$$\sum_{I\subseteq[n]} \mu(x_I) = 1$$

$$\sum_{I\subseteq[n]} \mu(x_I) Z_I^{t+1}(Z_I^{t+1})^\top \succeq 0$$

$$\sum_{I\subseteq[n]} \mu(x_I) g_i(x_I) Z_I^{t}(Z_I^{t})^\top \succeq 0 \text{ for } i \in [m]$$

$$x_I = \begin{cases} 1 \text{ if } i \in I \\ 0 \text{ otherwise} \end{cases}$$

$$Z_I^t[J] = \begin{cases} 1 \text{ if } J \in I, \quad |J|, |I| \leqslant t \\ 0 \text{ otherwise} \end{cases}$$

$$\tilde{\mathbb{E}}f = \sum_{I\in[n]} \mu(x_I) f(x_I)$$

SIMPLIFYING THE PSDNESS CONDITIONS

In general the PSDness is difficult to analyse

Idea: simplify by setting $\mu_k = \mu(x_I)$ *for every* I *with* $|I| = k$*.*

"Every solution of the same size gets the same value".

SoS constraint 2. simplifies to:

$$\sum_{I \subseteq [n]} \mu(x_I) Z_I^t (Z_I^t)^\top = \sum_{k=0}^{n} \mu_k \sum_{I \subseteq [n] : |I| = k} Z_I^t (Z_I^t)^\top$$

If constraints g symmetric s.t. $g_k = g(x_I)$ for every I with $|I| = k$, SoS constraint 3. simplifies to

$$\sum_{I \subseteq [n]} \mu(x_I) g(x_I) Z_I^t (Z_I^t)^\top = \sum_{k=0}^{n} \mu_k g_k \sum_{I \subseteq [n] : |I| = k} Z_I^t (Z_I^t)^\top$$

MAIN RESULT

Theorem

For any set of values $\{z_k \in \mathbb{R} : k = 0, ..., n\}$*, we have*

$$\sum_{k=0}^{n} z_k \sum_{I \subseteq [n] : |I| = k} Z_I^t (Z_I^t)^\top \succeq 0$$

if and only if

$$\sum_{k=h}^{n-h} \binom{n}{k} z_k P_h(k) \geqslant 0 \ \forall h \in \{0, ..., t\}$$

for every "well behaved" univariate polynomial $P_h(k)$*. In particular*

1. $\deg(P_h) = 2t$*.*
2. $P_h(k) = 0$ *for* $k \in \{0, \ldots, h-1\} \cup \{n-h+1, \ldots, n\}$*,*
3. $P_h(k) \geqslant 0$ *for* $k \in [h-1, \ldots, n-h+1]$*.*

Main Theorem - Discussion

Symmetry in the variables $\mu(x_I)$ allows us to find eigenvectors with many repeated entries $\to$ simplified analysis.

The derivation is mainly manipulating the quadratic form

$$v^\top \left(\sum_{I \subseteq [n] : |I| = k} Z_I^t (Z_I^t)^\top \right) v.$$

We identify all possible forms of the eigenvalues of the matrix and propose to test several candidates.

Recently, a similar result was independently obtained on the dual side by Blekherman.

Main Theorem - Related Results

Theorem (Blekherman)

Let $f : \{0,1\}^n \to \mathbb{R}_+$ be a symmetric non-negative real-valued boolean function and $\tilde{f}$ a univariate polynomial such that $f(x_1, \ldots, x_n) = \tilde{f}(x_1 + \ldots + x_n)$. If f can be written as the sum of squares of n-variate polynomials of degree $d \leqslant n/2$, then we can write

$$\tilde{f}(z) = q_d(z) + z(n-z)q_{d-1}(z) + z(z-1)(n-z)(n-1-z)q_{d-2}(z) + \ldots + z(z-1)\cdots(z-d+1)(n-z)(n-1-z)\cdots(n-d+1-z)q_0(z)$$

where each $q_t(z)$ is a univariate SoS polynomial with $deg(q_t) \leqslant 2t$.

RHS of the SoS representation of univariate polynomial $\tilde{f}$ can be written as a combination of polynomials P_h and vice versa.

MAX CUT IN THE COMPLETE GRAPH

Input: a graph $G = (V, E)$,
Output: a set of vertices $S \subseteq V$ such that number of edges between S and $V \setminus S$ is maximized.

$$f(x) = \left(\|x\|_1 - \left\lfloor \frac{n}{2} \right\rfloor\right)\left(\|x\|_1 - \left\lceil \frac{n}{2} \right\rceil\right) \geqslant 0 \qquad x \in \{0,1\}^n$$

The SoS rank is at least $\lfloor \frac{n}{2} \rfloor$ when n is odd (Grigoriev; Laurent).
And at most $\lceil \frac{n}{2} \rceil$ (Fawzi, Saunderson, Parrilo).

Used for example by Lee, Raghavendra and Steurer to derive extension complexities for max CSPs.

Question (O'Donnell): Is there a simple proof?

MAX CUT IN THE COMPLETE GRAPH

Consider the symmetric solution: $\mu_k = (n+1)\binom{n/2}{n+1}\frac{(-1)^{n-k}}{n/2-k}$.

- Recall: μ feasible for SoS if

$$\sum_{k=0}^{n}\binom{n}{k}\mu_k P_h(k) \geqslant 0$$

where P_h is of degree $2t$ and $P_h(n/2) \geqslant 0$.

Polynomial rem. thm.: $P_h(k) = (n/2-k)Q(k) + P_h(n/2)$.

$$\sum_{k=0}^{n}\binom{n}{k}\mu_k P_h(k) = \underbrace{\sum_{k=0}^{n}\binom{n}{k}\mu_k(n/2-k)Q(k)}_{=0} + P_h(n/2)\underbrace{\sum_{k=0}^{n}\binom{n}{k}\mu_k}_{=1} \geqslant 0$$

BPOP

BPOP: $\min_{x\in\{0,1\}^n} f(x)$ where $\deg(f) = r$.

For $r = 2$:

At least $\lfloor \frac{n}{2} \rfloor$ levels required for exact solution (Laurent).

- Follows by showing that the SoS relaxation of the function

$$f(x) = \left(\|x\|_1 - \left\lfloor \frac{n}{2} \right\rfloor\right)\left(\|x\|_1 - \left\lceil \frac{n}{2} \right\rceil\right)$$

has a strictly negative minimum over $\{0,1\}^n$.

At most $\lceil \frac{n}{2} \rceil$ levels are sufficient (Fawzi, Saunderson, Parrilo).

For $r > 2$:

At most $\lceil \frac{n+r-1}{2} \rceil$ levels suffice (Sakaue, Takeda, Kim and Ito).

- Sakaue et al. - numerical evidence that their bound is tight.

Our lower bound

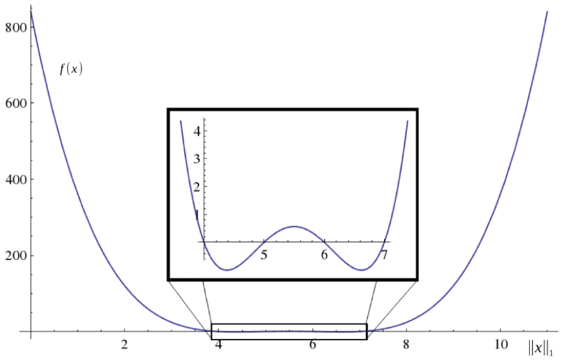
Fix n odd, let $r = 2d$ and consider the problem

$$\min_{x \in \{0,1\}^n} f_d(x) = \left(\|x\|_1 - \left\lfloor \frac{n}{2} \right\rfloor + d - 1 \right)^{\underline{2d}}$$

E.g. for n = 11

- $f_1(x) = (\|x\|_1 - 5)(\|x\|_1 - 6)$
- $f_2(x) = (\|x\|_1 - 4)(\|x\|_1 - 5)(\|x\|_1 - 6)(\|x\|_1 - 7)$

EXAMPLE: $f_2(x)$ WITH $n = 11$



OUR LOWER BOUND

Fix n odd, let $r = 2d$ and consider the problem

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Clearly the minimum of f_d over $\{0,1\}^n$ is 0.

We show: the minimum of the SoS relaxation of f_d is strictly negative at level $\lceil \frac{n+2d-1}{2} \rceil - 1$.

OUR LOWER BOUND - SKETCH OF PROOF

(Goal:)

the min of the SoS of f_d is strictly negative at level $\lceil \frac{n+2d-1}{2} \rceil - 1$.

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1. Find a candidate solution μ to the SoS relaxation.
   - $f_d(x)$ depends only on $\|x\|_1 \rightarrow$ choose μ_k that depends only on $|I| = k$.
2. Show that $\sum_{k=0}^{n} \mu_k \sum_{\substack{I \subseteq [n], \\ |I|=k}} Z_I Z_I^\top \succeq 0$ (feasibility).
   - μ_k symmetric $\rightarrow$ application of the symmetry theorem.
3. Show that $\sum_{k=0}^{n} \binom{n}{k} f_d(x_k) \mu_k < 0$ (objective strictly negative).
   - μ symmetric $\rightarrow$ reduces to a certain hypergeometric sum that can be computed in closed form using standard techniques.

SET COVER

$$SC: \quad \min \sum_{i\in[n]} x_i \quad \text{s.t.} \sum_{i\in[n]\setminus\{j\}} x_i \geqslant 1 \quad \forall j \in [n],\ x \in \{0,1\}^n$$

SA - requires at least $n-3$ (Bienstock & Zuckerberg)
i.e. to refute the inequality $\sum_{i=1}^n x_i < 2$

Bienstock & Zuckerberg conjecture: SoS needs at least $n/4$.

Our result:
$\log^{1-\epsilon} n$ levels of the SoS does not refute $\sum_{i=1}^n x_i \leqslant 1 + o(1)$.

Alternatively: SoS has an integrality gap of $2 - o(1)$ for MIN KNAPSACK strengthened with cover inequalities.

DETECTING EMPTY INTEGRAL HULL

Consider the system

$$EIH: \sum_{r \in R} x_r + \sum_{r \in [n] \setminus R} (1 - x_r) \geqslant \frac{1}{B} \quad \text{for all } R \subseteq [n],\ x \in \{0,1\}^n$$

Theorem (K, Leppänen, Mastrolilli'15)

For $B \geqslant 2^{n+1}$, SoS requires n levels.

What about other values of B?

EMPTY INTEGRAL HULL

$$EIH: \sum_{r \in R} x_r + \sum_{r \in [n] \setminus R} (1 - x_r) \geqslant \frac{1}{B} \quad \text{for all } R \subseteq [n],\ x \in \{0,1\}^n$$

For $B = 2$:

- Lovász-Schrijver - rank n (Goemans, Tunçel),
- Lovász-Schrijver with Chvátal or Gomory mixed integer cuts - rank n (Cook, Dash; Cornuéjols, Li),
- Sherali-Adams - rank n (Laurent),

Laurent conjecture: the SoS rank is $n - 1$

Our result: $\Omega(\sqrt{n}) \leqslant$ SoS rank $\leqslant n - \Omega(n^{1/3})$.

DETECTING EIH - PROOF SKETCH

By symmetry, it is enough to consider the solution $\mu(x_I) = \frac{1}{2^n}$.

By symmetry, it is enough to consider the constraint for $R = [n]$:

$$\frac{1}{2^n} \sum_{k=h}^{n-h} \binom{n}{k} \left(k - \frac{1}{2}\right) P_h(k) \geqslant 0 \ \forall h \in \{0, ..., t\} \tag{1}$$

LB: $\Omega(\sqrt{n}) \leqslant$ *SoS rank.*

We express the generic polynomial P_h in root form and argue that if the level is small (1) is always satisfied.

UB: SoS rank $\leqslant n - \Omega(n^{1/3})$

We give one polynomial $P_0(k) = \prod_{i=1}^{t}(n-k-i+1)^2$ for which we show that (1) is never satisfied when the level is large.

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$\mathbb{SOS}^{2d}$ - TWO OPTIONS

Primal

$$\max_{\lambda \in \mathbb{R}} \quad \lambda$$
$$s.t. \quad f - \lambda \in \Sigma_{\mathcal{G}}^{2d}$$

Dual

$$\min_{\tilde{\mathbb{E}}:\, \mathbb{R}[x]_{(2d+\deg \mathcal{G})} \to \mathbb{R}} \quad \tilde{\mathbb{E}}[f(x)]$$
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DIFFICULTIES WITH PROVING IG

PSDness of very large metrices

$$M_{g_0(\cdot)}, \ldots, M_{g_m(\cdot)} \in \mathbb{R}^{\binom{n+d}{d} \times \binom{n+d}{d}}$$

For a given instance one has to provide

solution vector $\tilde{\mathbb{E}} \in \mathbb{R}^{\binom{n+d}{d}}$

such that:

- attains superoptimal objective value: $\tilde{\mathbb{E}}[f(x)] < \min_{x \in \mathcal{G}_+} f(x)$
- $M_{g_0(x)}, \ldots, M_{g_m(x)} \succeq 0$.

SoS rank

SoS has rank d for EIH if there exist s_0 and s_I

$$EIH: \sum_{r\in R} x_r + \sum_{r\in[n]\setminus R}(1-x_r) \geqslant \frac{1}{B} \quad \text{for all } R\subseteq[n],\ x\in\{0,1\}^n$$

$$-1 = \underbrace{s_0}_{\text{deg } 2d+2} + \sum_{I\subseteq[n]} \underbrace{s_I(x)}_{\text{deg } 2d}\left(\sum_{i\in[n]\setminus I} x_i + \sum_{j\in I}(1-x_j) - 1/B\right)$$

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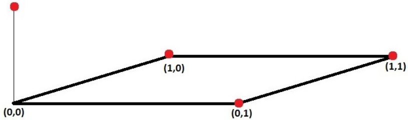
PROOF THAT DEGREE n IS ENOUGH

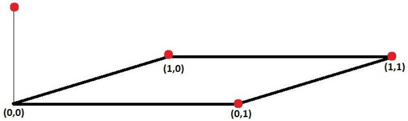


SoS certificate of degree 2 :

$$g(x) = \underbrace{\left(\sqrt{g(0,0)} \cdot (1-x_1)(1-x_2)\right)^2}_{s_0}$$

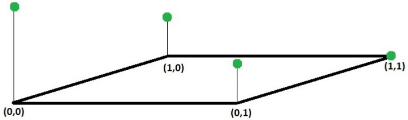
$$f(x) = \underbrace{\sum_{I \subseteq [n]} \left(\sqrt{f(x_I)} \cdot \delta_I(x)\right)^2}_{s_0}$$

PROOF THAT DEGREE n IS ENOUGH

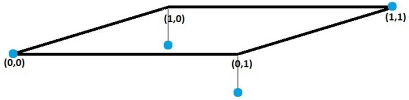


Is SoS certificate of degree 1 possible?

Consider s_0 of degree 1:



We require $f - s_0 \geqslant 0$ but we have:



ATTACK PLAN

1. Prove upper bounds on how spiky can a SoS of degree d be.

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2. Perform a boolean analysis on the function f and find vertices that cannot be satisfied.

HOW SPIKY CAN BE SOS OF DEGREE d

Lemma

For every function $f : \{0,1\}^n \to \mathbb{R}$ of degree at most d and every subset $J \subseteq [n]$ the following holds:

$$\sum_{S \subseteq [n]} f^2(x_S) \geqslant \frac{2^n}{\sum_{i=0}^{d} \binom{n}{i}} f^2(x_J).$$

Moreover, for every $J \subseteq [n]$ there exists a degree d polynomial such that the above inequality is satisfied with equality.

Moreover holds if we put SoS instead of f:

$\sum_i h_i^2$ s.t. for every i, $h_i : \{0,1\}^n \to \mathbb{R}$ is of degree at most d

Main Theorem

Theorem

There is no degree d certificate for f over system $\mathcal{G}$, if for every $c_J \geqslant 1$, for every J such that $x_J \in \mathcal{H}^-(f)$, the following holds

$$\sum_{x_J \in \mathcal{H}^+(f)} f(x_J) + \sum_{x_J \in \mathcal{H}^-(f)} f(x_J)(1 - c_J) < \left(\frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} - 1 \right) \sum_{x_I \in \mathcal{H}^-(f)} c_I \frac{f(x_I)}{g_I(x_I)} \cdot \min_{\substack{x_J \in \mathcal{H} \\ J \neq I}} g_I(x_J)$$

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APPLICATION TO EIH PROBLEM

$$-1 = \underbrace{s_0}_{\text{deg } 2d+2} + \sum_{I \subseteq [n]} \underbrace{s_I(x)}_{\text{deg } 2d} \left(\sum_{i \in [n] \setminus I} x_i + \sum_{j \in I} (1 - x_j) - 1/B \right)$$

From the main Theorem, no degree d SoS proof if

$$\sum_{x_J \in \mathcal{H}^+(f)} f(x_J) + \sum_{x_J \in \mathcal{H}^-(f)} f(x_J)(1 - c_J) < \left(\frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} - 1 \right) \sum_{x_I \in \mathcal{H}^-(f)} c_I \frac{f(x_I)}{g_I(x_I)} \cdot \min_{\substack{x_J \in \mathcal{H} \\ J \neq I}} g_I(x_J)$$

$$\Downarrow$$

$$(-1) \sum_{I \in [n]} (1 - c_J) < \left(\frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} - 1 \right) \sum_{I \in [n]} c_I \frac{-1}{-1/B} \cdot (1 - 1/B)$$

For $c = 1/2^n \sum_{I \subset [n]} c_I$

$$\frac{c-1}{c} < \left(\frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} - 1 \right) (B - 1) \qquad \Rightarrow \qquad \frac{1}{B-1} < \left(\frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} - 1 \right)$$

Application to EIH problem

Lemma

SoS rank is lower and upper bounded by minimum d that satisfies respectively:

$$\frac{B}{B-1} \geqslant \frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}} \quad \text{and} \quad \frac{Bn}{Bn-1} > \frac{2^n}{\sum_{k=0}^{d} \binom{n}{k}}$$

For $B = 2$**:** $\lceil \frac{n}{2} \rceil \leqslant SoS - rank \leqslant \lceil \frac{n}{2} + \sqrt{n \log 2n} \rceil$

APPROXIMATION THEORY

Moreover, we use the following result from the literature:

Theorem (Paturi, Sherstov, de Wolf)

For every constant $2^{-n} \leqslant c < 1/2$ the minimum degree of a real polynomial that approximates a NOR boolean function in ℓ_∞-norm within an error c is $\Theta(\sqrt{n} + \sqrt{n \log 1/c})$.

MIN KNAPSACK

$$MK: \quad \min \sum_{i \in [n]} x_i \quad \text{s.t.} \sum_{i \in [n]} x_i \geqslant \frac{1}{P}, \quad x \in \{0,1\}^n$$

For $P = 2$:

- Lovász-Schrijver - rank n (Cook, Dash),
- Sherali-Adams - rank n (Laurent),
- SoS - rank 2 for $n = 2$ (Laurent)
- SoS -rank $\geqslant \Omega(\log^{1-\epsilon} n)$, for $\epsilon > 0$.

Laurent question: what is the SoS rank for MK *Problem.*

Our result: $\Omega(\sqrt{n}) \leqslant SoS - rank \leqslant \lceil \frac{n+4\lceil \sqrt{n} \rceil}{2} \rceil$

Application to MK problem

$$\sum_{i\in[n]} x_i - 1 = \underbrace{s_0}_{\text{deg } 2d+2} + \underbrace{s_1(x)}_{\text{deg } 2d} \left(\sum_{i\in[n]} x_i - 1/P\right)$$

Degree d SoS necessarily has to satisfy

$$s_1(0,\ldots,0) \geqslant \frac{-1}{-\frac{1}{P}} = P \quad \text{and} \quad s_1(x) \leqslant \frac{\sum_{i=1}^{n} x_i - 1}{\sum_{i=1}^{x} x_i - \frac{1}{P}} \leqslant 1 \text{ for } {}_{x\neq(0,\ldots,0)}^{x\in\{0,1\}^n}$$

Lemma

The SoS rank lower bound is at least $\Omega(\sqrt{n} + \sqrt{n \log P})$.

$\bar{s}(x) := \frac{s(x)}{P}$ would approximates NOR with l_∞-norm within an error 1/P.

SET COVER

$$SC: \quad \min \sum_{i\in[n]} x_i \quad \text{s.t.} \sum_{i\in[n]\setminus\{j\}} x_i \geqslant 1 \quad \forall j \in [n],\ x \in \{0,1\}^n$$

- SA - rank at least $n-2$ (Bienstock & Zuckerberg)

Bienstock & Zuckerberg question: what is the SoS rank for SC?

Bienstock & Zuckerberg conjecture: SoS rank is $\geqslant n/4$.

Our result: $\Omega(\sqrt{n}) \leqslant SoS-rank$

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APPLICATION TO SC PROBLEM

$$\sum_{i\in[n]} x_i - 2 = \underbrace{s_0}_{deg\ 2d+2} + \sum_{j\in[n]} \underbrace{s_j(x)}_{deg\ 2d} \left(\sum_{j\neq i\in[n]} x_i - 1 \right)$$

Let $s = \sum_{j\in[n]} s_j$. Degree d SoS necessarily has to satisfy

$$s(0,\ldots,0) \geqslant \frac{-1}{-\frac{1}{2}} = 2 \quad \text{and} \quad s(x) \leqslant \frac{\sum_{i=1}^{n} x_i - 1}{\sum_{i=1}^{x} x_i - \frac{1}{2}} \leqslant 1 \text{ for } {}_{\substack{x\in\{0,1\}^n \\ |x|=3k,\ k\in[n/3]}}$$

Lemma

The SoS rank lower bound is at least $\Omega(\sqrt{n})$.

$\bar{s}(x) := \frac{s(x)}{3}$ would approximates NOR with l_∞-norm within an 1/3 over the hypercube $\{0,1\}^{n/3}$.

Thank you