Non-Commutative Polynomial Optimzation

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Warm Up

What is a non-commutative polynomial?

- matrix polynomial $M_2(\mathbb{R}[x])$

$$\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} x^2 + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x^2 + 2x + 1 & 2x^2 \\ -2x^2 + x - 1 & 3x^2 + x \end{pmatrix}$$

- free polynomial $\mathbb{R}\langle X, Y \rangle$

$$X^2 + XY - YX + Y^2 \neq X^2 + Y^2$$

- trace polynomial $\mathbb{R}[\operatorname{Tr}(X^k) : k \in \mathbb{N}]\langle X \rangle$

$$\operatorname{Tr}(X)X^2 + 2\operatorname{Tr}(X^2)X - X^2 + 2$$

- ...

Warm Up

What should X represent?

- scalars
- matrices of arbitrary size
- (bounded) operators: symmetric, anti-symmetric
- differential operators, Weyl-operators
- matrices of fixed size
- ...

This talk

Replace scalars by symmetric matrices/operators $\rightarrow$ free polynomials

RAG and POP basics

Polynomial Optimization

- $f \in \mathbb{R}[\underline{X}]$ polynomial in commuting variables
- $g_0 = 1, g_1, \ldots, g_r \in \mathbb{R}[\underline{X}]$ defining a semi-algebraic set:

$$K = \{\underline{a} \in \mathbb{R}^n \mid g_0(\underline{a}) \geq 0, \ldots, g_r(\underline{a}) \geq 0\}$$

- Want to minimize f over K

$$\begin{aligned} f_* &= \inf f(\underline{a}) && \text{s.t. } \underline{a} \in K \\ &= \sup a \in \mathbb{R} && \text{s.t. } f - a \geq 0 \text{ on } K \end{aligned}$$

- NP-hard

RAG and POP basics

RAG helps

$$f_* = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \geq 0 \text{ on } K$$

NP-hard ☹

- $M(g) := \{p = \sum_j h_j^2 g_{i_j} \text{ for some } h_j \in \mathbb{R}[\underline{X}]\}$
- sos relaxation (Lasserre, Parrilo)

$$f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)$$

RAG and POP basics

RAG helps

$$f_* = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \geq 0 \text{ on } K$$

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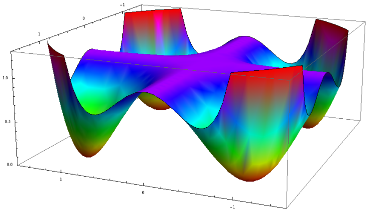
- sos relaxation (Lasserre, Parrilo)

$$f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)$$

- f_{sos} is always a lower bound but might be strict



$x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$

RAG and POP basics

SOS hierarchy

- $M(g)_t := \{p = \sum_j h_j^2 g_{i_j} \text{ for some } h_j \in \mathbb{R}[\underline{X}]_t\}$
- sos hierarchy (Lasserre, Parrilo)

$$\boxed{f_t = \sup a \in \mathbb{R} \quad \text{s.t.}\ f - a \in M(g)_t}$$

SDP ☺

- SDP due to the Gram matrix method:

$$f \text{ sos} \iff \exists\, G \succeq 0 : f(x) = [x]^T G[x]$$

RAG and POP basics

SOS hierarchy

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SDP ☺

- SDP due to the Gram matrix method:

$$f \text{ sos} \iff \exists\, G \succeq 0 : f(x) = [x]^T G[x]$$

- We have
  - $f_t \leq f_{t+1} \leq f_{sos} \leq f_*$
  - f_t converges to f_{sos} as $t \to \infty$
  - Putinar: If $M(g)$ is archimedean: $f_{sos} = f_*$

RAG and POP basics

Moment problem

$$\boxed{f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)}$$

- dual problem

$$\boxed{f_{mom} = \inf L(f) \quad \text{s.t. } L \in \mathbb{R}[\underline{x}]^{\vee}, L \geq 0 \text{ on } M(g)}$$

- This is an SDP (up to degree bounds), using moment matrices

RAG and POP basics

Moment problem

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- dual problem

$$f_{mom} = \inf L(f) \quad \text{s.t. } L \in \mathbb{R}[\underline{x}]^{\vee}, L \geq 0 \text{ on } M(g)$$

- This is an SDP (up to degree bounds), using moment matrices
- If optimizing L in f_{mom} has a ▸ moment representation then

$$f_* \geq f_{sos} \geq f_{mom} = L(f) \geq f_*$$

- Moment representation implies exactness of relaxation

Theorem (Curto, Fialkow)

If the moment matrix of L is flat, then L has a moment representation.

NC-RAG and NC-POP

Free Polynomials

- Want to replace scalar variables by matrices/operators
- Free algebra $\mathbb{R}\langle \underline{X} \rangle$ with non-commuting variables $X_1, \ldots, X_n$
- Polynomial

$$f = \sum_{w} f_w w$$

- Let $\underline{A} \in (\mathcal{S}^d)^n$: $f(\underline{A}) = f_1 I_d + f_{X_1} A_1 + f_{X_2 X_1} A_2 A_1 \ldots$

NC-RAG and NC-POP

Free Polynomials

- Want to replace scalar variables by matrices/operators
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- Polynomial
$$f = \sum_w f_w w$$
- Let $\underline{A} \in (\mathcal{S}^d)^n$: $f(\underline{A}) = f_1 I_d + f_{X_1} A_1 + f_{X_2 X_1} A_2 A_1 \ldots$
- Add involution $*$ on $\mathbb{R}\langle \underline{X} \rangle$
  - fixes $\mathbb{R}$ and $\{X_1, \ldots, X_n\}$ pointwise
  - $X_i^* = X_i$
- Consequence
$$f^* f(\underline{A}) = f(\underline{A})^T f(\underline{A}) \succeq 0$$

NC-RAG and NC-POP

Free Polynomial Optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$
- $g_0 = 1, g_1, \ldots, g_r \in \mathbb{R}\langle \underline{X} \rangle$ defining a semi-algebraic set:

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- Want to minimize f over K

$$\boxed{f_* = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \geq 0 \text{ on } K}$$

What does $f \geq 0$ mean?

NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

$$\boxed{f_{nc} = \inf \operatorname{eig}(f(\underline{A})) \quad \text{s.t. } \underline{A} \in K}$$

NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

$$\boxed{f_{nc} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \succeq 0 \text{ on } K}$$

- sos relaxation

  $M_{nc}(g) := \{p = \sum_j h_j^* g_{i_j} h_j \text{ for some } h_j \in \mathbb{R}\langle \underline{X} \rangle\}$

$$\boxed{f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M_{nc}(g)}$$

- Fact: $f_{sos} \leq f_{nc}$

NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

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$$\boxed{f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M_{nc}(g)}$$

- Fact: $f_{sos} \leq f_{nc}$
- Observation: Gram matrix method still works
  - Checking if $f = \sum_i h_i^* h_i$ is an SDP
  - Checking if $f = \sum_j h_j^* g_{i_j} h_j$ (with degree bounds) is an SDP

Example

NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

$$f_{nc} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \succeq 0 \text{ on } K$$

- $M_{nc}(g)_t := \{p = \sum_j h_j^* g_{i_j} h_j \text{ for some } h_j \in \mathbb{R}\langle \underline{X} \rangle_t\}$
- sos hierarchy (Navascués, Pironio, Acín)

$$f_t = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M_{nc}(g)_t$$

SDP ☺

NC-RAG and NC-POP

Eigenvalue optimization

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SDP ☺

- $f_t \leq f_{t+1} \leq f_{sos} \leq f_{nc}$ but inequalities might be strict
- f_t converges to f_{sos} as $t \to \infty$
- Helton et al.: If $M_{nc}(g)$ archimedean: $f_{sos} = f_{nc}$

NC-RAG and NC-POP

NC Moment problem

$$\boxed{f_{mom} = \inf L(f) \quad \text{s.t.}\ L \in \mathbb{R}\langle \underline{X} \rangle^{\vee}, L \geq 0 \text{ on } M_{nc}(g)}$$

- This is an SDP (up to degree bounds), using ▸ moment matrices

NC-RAG and NC-POP

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NC moment problem

For which linear form $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ exists a (finite dimensional) Hilbert space H, a unit vector $\phi \in H$ and a $*$-representation π on $B(H)$ such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \langle \pi(f)\phi, \phi \rangle?$$

- Moment representation implies exactness of relaxation

Theorem (Klep et al.)

If the moment matrix of L is flat, then L has a moment representation. In this case we can also extract a fin. dim. optimizer for f.

NC-RAG and NC-POP

Eigenvalue optimization: bonus

- Helton/McCullough: $f \succeq 0 \iff f$ sos

NC-RAG and NC-POP

Eigenvalue optimization: bonus

- Helton/McCullough: $f \succeq 0 \iff f$ sos
  - proof idea: construct (GNS) a moment representation
    - Assume $f \succeq 0$ but not sos: separating linear form L
    - induces a positive semidefinite bilinear form
    - Hilbert space $\mathcal{H}$ as completion of $\mathbb{R}\langle \underline{X} \rangle / N$ with $N = \{g \in \mathbb{R}\langle \underline{X} \rangle \mid L(g^* g) = 0\}$
    - moment representation via $\hat{X}_i : \mathcal{H} \to \mathcal{H}, p \mapsto X_i p$
    - $L(p) = \langle \hat{p} 1, 1 \rangle_{\mathcal{H}} = \langle p(\underline{A}) 1, 1 \rangle$ for some representations A_i of $\hat{X}_i$.
- Remark: positivity on matrices of a fixed size is sufficient

NC-RAG and NC-POP

Eigenvalue optimization: bonus

- Helton/McCullough: $f \succeq 0 \iff f$ sos
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- Remark: positivity on matrices of a fixed size is sufficient

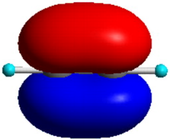
- If K is NC-cube or NC-ball we need just one step in the hierarchy
  - proof idea: Construct a flat moment matrix
- If K is NC-convex, positivity on matrices of a fixed size is sufficient

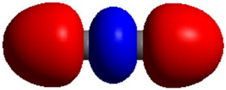
Application: Quantum Chemistry

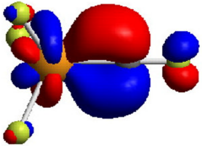
Compute ground state energy of atoms

- Molecule of N electrons that can occupy M orbitals
- Each orbital associated with creation/anihilation operators $a_i^\dagger, a_i$
- Pairwise interaction described by parameters h_{ijkl}

$$\begin{aligned}
\min_{(a,a^\dagger,\varphi)} \quad & \left\langle \varphi, \sum_{ijkl} h_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \varphi \right\rangle \\
\text{s.t.} \quad & \|\varphi\| = 1 \\
& \{a_i^\dagger, a_j\} := a_i^\dagger a_j + a_j a_i^\dagger = \delta_{ij} \\
& \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \\
& \Big(\sum_i a_i^\dagger a_i - N\Big)\varphi = 0
\end{aligned}$$

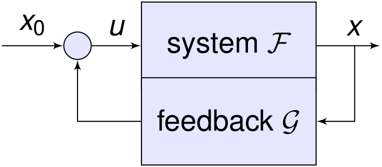






Application: Systems Control

- Linear closed loop system with unknown feedback $\mathcal{G}$



Math. System

$$\dot{\vec{x}}(t) = \mathcal{A}\vec{x}(t) + \mathcal{B}\vec{u},$$
$$\vec{u}(t) = \mathcal{C}\vec{x}(t)$$

- Goal Find $\mathcal{G}$ which stabilizes the system

Application: Systems Control

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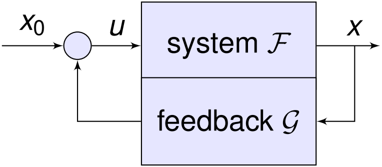

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Lyapunov1892

A system $\dot{x}(t) = Ax(t)$ is stable if there is a $P \succeq 0$ with $A^t P + PA \prec 0$

- Lyapunov's idea extends to our problem: Riccati equations

Application: Systems Control

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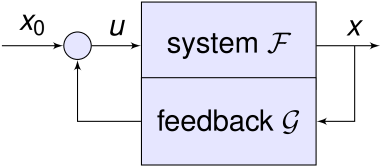

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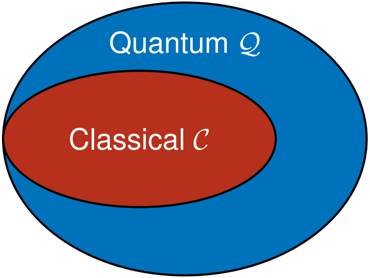
A system $\dot{x}(t) = Ax(t)$ is stable if there is a $P \succeq 0$ with $A^t P + PA \prec 0$

- Lyapunov's idea extends to our problem: Riccati equations
- Optimization problem is first a feasibility problem
- Can be refined by optimizing a specific singular value
- For a uniform strategy to get $\mathcal{G}$ we have to work free of dimensions

Application: Quantum correlations

- Entanglement is one of the key features in Quantum Information
- Bell '64:



- How to distinguish $\mathcal{C}$ and $\mathcal{Q}$?
- Bell-inequalities, e.g. $E_0F_0 + E_0F_1 + E_1F_0 - E_1F_1$

Basics of quantum theory

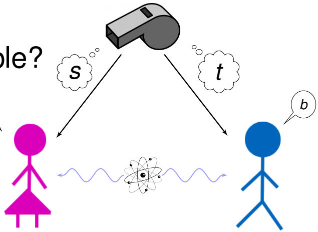
- A quantum system corresponds to a Hilbert space $\mathcal{H}$
- Its states are unit vectors on $\mathcal{H}$
- A state on a composite system is a unit vector ψ on a tensor Hilbert space, e.g. $\mathcal{H}_A \otimes \mathcal{H}_B$
- ψ is entangled if it is not a product state

$$\psi_A \otimes \psi_B \text{ with } \psi_A \in \mathcal{H}_A, \psi_B \in \mathcal{H}_B$$

- A state $\psi \in \mathcal{H}$ can be measured
  - outcomes $a \in A$
  - POVM: a family $\{E_a\}_{a \in A} \subseteq B(\mathcal{H})$ with $E_a \succeq 0$ and $\sum_{a \in A} E_a = 1$
  - probablity of getting outcome a is $p(a) = \psi^T E_a \psi$.

Nonlocal bipartite correlations

- Question sets S, T, Answer sets A, B
- No (classical) communication
- Which correlations $p(a, b \mid s, t)$ are possible?



- Nonlocal game: winning predicate $V : A \times B \times S \times T \to \{0, 1\}$
- Winning probability (value of the game)

$$\omega = \sup_p \sum_{s \in S, t \in T} \pi(s, t) \sum_{a \in A, b \in B} V(a, b; s, t) p(a, b | s, t)$$
$$= \sup_p \sum_{a,b,s,t} f_{abst} p(a, b \mid s, t)$$

Correlations

Classical strategy $\mathcal{C}$

Independent probability distributions $\{p_s^a\}_a$ and $\{p_t^b\}_b$:

$$p(a, b \mid s, t) = p_s^a \cdot p_t^b$$

shared randomness: allow convex combinations

$$\boxed{\omega = \sup_{(x,y)} \sum_{a,b,s,t} f_{abst}\, x_s^a\, y_t^b}$$

Correlations

Classical strategy $\mathcal{C}$

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Quantum strategy $\mathcal{Q}_c$

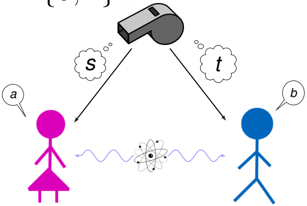
POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on a joint Hilbert space, but $[E_s^a, F_t^b] = 0$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \cdot F_t^b)\psi$$

$$\boxed{\omega = \sup_{(X,Y,\psi)} \sum_{a,b,s,t} f_{abst} \, X_s^a \, Y_t^b}$$

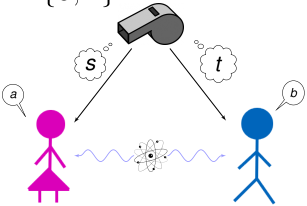
CHSH Game

- Questions $S = T = \{0, 1\}$, Answers $A = B = \{0, 1\}$
- Alice & Bob win, if $a + b \equiv st \mod 2$



CHSH Game

- Questions $S = T = \{0, 1\}$, Answers $A = B = \{0, 1\}$
- Alice & Bob win, if $a + b \equiv st \mod 2$
- $\omega_{\mathcal{C}} = \frac{3}{4}$
- $\omega_{\mathcal{Q}_c} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.854$



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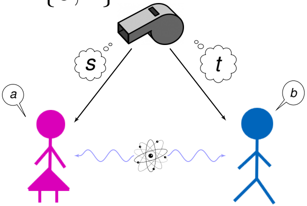

- lower bounds by brute force
- upper bounds via SOS hierarchies of operator formulation:
- 2 measurements with 2 outcomes each: $E_s^0, E_s^1, F_t^0, F_t^1$
- Setting $E_s := E_s^0 - E_s^1$, $F_t := F_t^0 - F_t^1$: CHSH inequality

$$f_{CHSH} := E_0F_0 + E_0F_1 + E_1F_0 - E_1F_1$$

- Optimizing f_{CHSH} over variants of $\mathcal{C}, \mathcal{Q}_c$ give $\omega_{\mathcal{C}}, \omega_{\mathcal{Q}_c}$

More correlations

Quantum strategy $\mathcal{Q}_c$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on a joint Hilbert space, but $[E_s^a, F_t^b] = 0$:

$$p(a, b \mid s, t) = \psi^T(E_s^a \cdot F_t^b)\psi$$

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Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \otimes F_t^b) \psi$$

More correlations

Quantum strategy $\mathcal{Q}_c$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on a joint Hilbert space, but $[E_s^a, F_t^b] = 0$:

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Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \otimes F_t^b) \psi$$

- locality: $(E_s^a \otimes 1)(1 \otimes F_t^b) = (1 \otimes F_t^b)(E_s^a \otimes 1)$
- If $\psi = \psi_A \otimes \psi_B$ then we have classical correlation

Fact

$$\mathcal{C} \subseteq \mathcal{Q} \subseteq \mathrm{cl}(\mathcal{Q}) \subseteq \mathcal{Q}_c$$

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Theorem (Ji, Natarajan, Vidick, Wright, Yuen,'20)

$\mathrm{cl}(\mathcal{Q}) \neq \mathcal{Q}_c$

- Ozawa: (strong) Tsirelson problem $\iff$ Connes conjecture

Connes embedding conjecture

Connes embedding conjecture

If ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{F}$ is a II_1 factor with separable predual, then $\mathcal{F}$ can be embedded into the ultrapower $\mathcal{R}^\omega$.

- $\mathcal{F}$ is a II_1 factor if $\mathcal{F}$ is a subsalgebra of $B(\mathcal{H})$ for an infinite dimensional Hilbert space $\mathcal{H}$ and allows for a finite tracial state
- $\mathcal{R}$ is the hyperfinite II_1 factor, i.e. it can be constructed as limit of matrix algebras
- $\mathcal{F}$ embeds into $\mathcal{R}^\omega$ iff it allows matricial microstates, i.e. tracial moments can be approximated by matricial tracial moments:
  Let $X = \{A_1, \ldots, A_n\} \subseteq \mathcal{F}_{sa}$ be finite, then $\forall\, k \in \mathbb{N}, \forall\, \varepsilon > 0$
  $\exists\, s \in \mathbb{N}, \exists B_1, \ldots, B_n \in M_s(\mathbb{C}) : |\tau(A_{i_1} \ldots A_{i_k}) - \mathrm{Tr}(B_{i_1} \ldots B_{i_k})| < \varepsilon.$

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The conjecture is false

Connes and NC RAG

- Let $f \in \mathbb{R}\langle \underline{X} \rangle_{sym}$
- $M_{tr} := \{\sum_i h_i^*(1 - X_i^2)h_i \mid h_i \in \mathbb{R}\langle \underline{X} \rangle\} + [\mathbb{R}\langle \underline{X} \rangle, \mathbb{R}\langle \underline{X} \rangle]$
- $K = \{\underline{A} \mid A_i \subseteq N, N$ finite vN algebra $: \mathbf{1} - A_i^2 \succeq 0$ for all $i \in [n]\}$.

Theorem (Klep, Schweighofer)

The following are equivalent

1. f trace-positive on K,
2. $\forall \varepsilon > 0 : f + \varepsilon \in M_{tr}$.

Theorem (Klep, Schweighofer, (& B., Dykema))

Connes' conjecture holds iff K can be replaced by

$$K_{fin} := \{\underline{A} \mid A_i \subseteq M_s(\mathbb{R}) \text{ for some } s \in \mathbb{N} : \mathbf{1} - A_i^2 \succeq 0 \text{ for all } i \in [n]\}.$$

Consequences

Operators on finite dimensional Hilbert spaces are not sufficient

- Tsirelson: There is a quantum correlation of the form $p(a, b \mid s, t) = \psi^T E_s^a \otimes F_t^b \psi$ which cannot be written as $p(a, b \mid s, t) = G_s^a \otimes H_t^b$ with commuting operators
- strong Tsirelson: There is a quantum correlation of the form $p(a, b \mid s, t) = \psi^T E_s^a \otimes F_t^b \psi$ which is even not a limit of quantum correlations in the commuting model
- Connes: There is a II_1 factor, where one cannot approximate its tracial moments by tracial moments using matrices
- NC RAG: There is a polynomial which is trace-positive on the matricial NC cube but not an element of the corresponding quadratic module

Consequence for tracial optimization

Operators on finite dimensional Hilbert spaces are not sufficient

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

$$\boxed{f_{tr} = \sup a \in \mathbb{R} \quad \text{s.t. } \operatorname{Tr}(f - a) \geq 0 \text{ on } K}$$

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- Choose K carefully:
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  - only matrices:
    it might be that $f_{sos} \neq f_{tr}$ even when M_{tr} is archimedean
  - also operators:
    it might be that $f_t = f_{tr}$ but there is no flat moment matrix at all
    (optimum attained only in infinite dimension)

Summary and Outlook

- Free polynomial optimization has a variety of applications:
  - quantum chemistry
  - systems control
  - nonlocal games
  - free LMIs: quantum channels
  - Weyl algebras: Schrödinger operators

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  - elaborate theory of polynomial identites
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  - find an optimality criterion without flat matrices
- Do research on trace-polynomials

$$\mathrm{Tr}(X)X^2 + 2\,\mathrm{Tr}(X^2)X - X^2 + 2$$

Thank you for your attention.

Gram matrix method

Example

$f = X^2Y^2 + Y^2X^2$

$$[X] = (X^2, XY, YX, Y^2)^T$$

$$[X]^*[X] = \begin{pmatrix} X^4 & X^3Y & X^2YX & X^2Y^2 \\ YX^3 & YX^2Y & YXYX & YXY^2 \\ XYX^2 & XYXY & XY^2X & XY^3 \\ Y^2X^2 & Y^2XY & Y^3X & Y^4 \end{pmatrix}$$

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$$g = 2X^4 - X^2YX - 2X^2Y^2 - XYX^2 + XY^2X - 2Y^2X^2 + 4Y^4$$

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$g = 2X^4 - X^2YX - 2X^2Y^2 - XYX^2 + XY^2X - 2Y^2X^2 + 4Y^4$

$$G = \begin{pmatrix} 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \succeq 0$$

$\rightarrow$ g is sos

▸ NC SOS

Classical multivariate moment problem

- Let $K \subseteq \mathbb{R}^n$ be closed.

Moment problem

Let $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear, $L(1) = 1$. Does there exist a probability measure μ with $\operatorname{supp}\mu \subseteq K$ such that for all $f \in \mathbb{R}[\underline{x}]$:

$$L(f) = \int f(\underline{a})\,\mathrm{d}\mu(\underline{a})?$$

Theorem (Riesz, Haviland)

Let $K \subseteq \mathbb{R}^n$ be non-empty and closed, $L \in \mathbb{R}[\underline{x}]^\vee$. There exists a measure μ supported on K such that

$$L(f) = \int f(\underline{a})\,\mathrm{d}\mu(\underline{a}) \text{ for all } f \in \mathbb{R}[\underline{x}]$$

if and only if $L(p) \geq 0$ for all $p \in \mathbb{R}[\underline{X}]$ that are **positive** on K.

SOS

Hankel matrices

- Associate to $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ the sesquilinear form

$$B_L : \mathbb{R}\langle \underline{X} \rangle \times \mathbb{R}\langle \underline{X} \rangle, (f, g) \mapsto L(f^* g).$$

- The representing matrix for B_L is its Hankel matrix

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- The representing matrix for B_L is its Hankel matrix

Definition

- The Hankel matrix $M(L)$, indexed by $u, v \in \langle \underline{X} \rangle$, is given by

$$M(L)_{u,v} := L(u^* v).$$

- The truncated Hankel matrix $M_k(L)$ of degree k is the submatrix of $M(L)$ indexed by $u, v \in \langle \underline{X} \rangle_k$.

One Hankel matrix

Example

Consider $\mathbb{R}\langle X, Y\rangle$ with basis $(1, X, Y, X^2, XY, YX, \dots)$

$$M(L) = \begin{bmatrix} L(1) & L(X) & L(Y) & L(X^2) & L(XY) & \dots \\ L(X) & L(X^2) & L(XY) & L(X^3) & L(X^2Y) & \dots \\ L(Y) & L(YX) & L(Y^2) & L(YX^2) & L(YXY) & \dots \\ L(X^2) & L(X^3) & L(X^2Y) & L(X^4) & L(X^3Y) & \dots \\ L(YX) & L(YX^2) & L(YXY) & L(YX^3) & L(YX^2Y) & \dots \\ L(XY) & L(XYX) & L(XY^2) & L(XYX^2) & L(XYXY) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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$$M_1(L) = \begin{bmatrix} L(1) & L(X) & L(Y) \\ L(X) & L(X^2) & L(XY) \\ L(Y) & L(YX) & L(Y^2) \end{bmatrix}$$

NC moment problem