Jordan symmetry reduction for conic optimization over the doubly nonnegative cone: theory and software

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Outline of the talk

- Conic optimization over the doubly nonnegative cone;
- Example: relaxation of quadratic assignment problems (QAPs);
- The Jordan symmetry reduction approach of Parrilo and Permenter for symmetric cones.

Reference:

F. N. Permenter and P. A. Parrilo. Dimension reduction for semidefinite programs via Jordan algebras. *Mathematical Programming*, 181, 51–84, 2020.

- Extension of Jordan reduction to the doubly nonnegative cone.
- Software and numerical examples (QAP).

Conic optimization over the doubly nonnegative cone

Conic optimization

General conic optimization problem:

$$\begin{aligned} \inf \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i \text{ for } i \in [m] \\ & X \in \mathcal{K} \end{aligned} \qquad = \begin{aligned} \inf \quad & \langle C, X \rangle \\ \text{s.t.} \quad & X \in X_0 + \mathcal{L} \\ & X \in \mathcal{K}, \end{aligned}$$

where

- $[m] = \{1, \dots, m\}$
- $\mathcal{K} \subseteq \mathcal{V}$ is a closed, convex cone in a real Hilbert space $\mathcal{V}$,
- $X_0 \in \mathcal{V}$ satisfies $\langle A_i, X_0 \rangle = b_i$ for all $i \in [m]$,
- $\mathcal{L} \subseteq \mathcal{V}$ is the nullspace of the operator $X \mapsto (\langle A_i, X \rangle)_{i=1}^m$.

Conic optimization over the doubly nonnegative cone

We are interested in the special case where:

- $\mathcal{V}$ is the space $\mathbb{S}^n$ of $n \times n$ symmetric matrices equipped with the Euclidean inner product;
- $\mathcal{K}$ is the cone of doubly nonnegative matrices $\mathcal{D}^n$ (positive semidefinite and entrywise nonnegative).

Such problems arise as convex relaxations of

- polynomial optimization problems with nonnegative variables;
- combinatorial optimization problems, like QAP.

Example: relaxation of quadratic assignment problem (QAPs)

General form of quadratic assigment problems (QAPs)

QAP in Koopmans-Beckmann form

$$QAP(A, B) = \min_{\varphi \in S_n} \sum_{i,j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)}$$

where

- $A = (a_{ij}), B = (b_{ij}) \in \mathbb{S}^n$
- S_n is the set of all permutations of n elements

Example: relaxation of QAP

Zhao, Karisch, Rendl, Wolkowicz (1998), Povh, Rendl (2009)

$$
\begin{aligned}
SDPQAP(A,B) := \min \ & \langle B \otimes A, Y \rangle \\
\text{s.t.} \ & \langle I \otimes E_{jj}, Y \rangle = 1 \text{ for } j = 1, \ldots, n, \\
& \langle E_{jj} \otimes I, Y \rangle = 1 \text{ for } j = 1, \ldots, n, \\
& \langle I \otimes (J - I) + (J - I) \otimes I, Y \rangle = 0, \\
& \langle J \otimes J, Y \rangle = n^2, \\
& Y \in \mathcal{D}^{n^2},
\end{aligned}
$$

where I the identity, and J the all-ones matrix, $E_{jj} = e_j e_j^\top$ for e_j the jth standard unit vector.

(Jordan) symmetry reduction

(Jordan) symmetry reduction

Conic optimization problem:

$$\inf \{\langle C, X\rangle \ : \ X \in X_0 + \mathcal{L},\ X \in \mathcal{K}\}.$$

- The idea of symmetry reduction is to restrict to a low dimensional subspace $S \subset \mathcal{V}$ that contains an optimal solution.
- Ideally, S also has additional structure, typically an algebra that can be further decomposed (block-diagonalization).
- In the approach of Parrilo-Permenter, we describe suitable S in terms of the orthogonal projection P_S onto S.

Constraint set invariance conditions (CSICs)

Notation: for any subspace S, $P_S : \mathcal{V} \to S$ denotes the orthogonal projection onto S.

Definition

We call S is an admissible subspace for the conic optimization problem

$$\inf \{\langle C, X\rangle \ : \ X \in X_0 + \mathcal{L},\ X \in \mathcal{K}\}.$$

if

(i) $P_S(\mathcal{K}) \subseteq \mathcal{K}$ (the projection is positive),

(ii) $P_S(X_0 + \mathcal{L}) \subseteq X_0 + \mathcal{L}$,

(iii) $P_S(C + \mathcal{L}^\perp) \subseteq C + \mathcal{L}^\perp$.

Key observation: if the conic optimization problem has an optimal solution, then there is an optimal solution in S.

Jordan reduction

Proposition (Permenter (2017))

If $\mathcal{V} = \mathbb{S}^n$, and $\mathcal{K} = \mathbb{S}^n_+$ (p.s.d. cone), then the following two conditions are equivalent:

1. $P_S(\mathcal{K}) \subseteq \mathcal{K}$;
2. $S \supseteq \{X^2 \mid X \in S\}$ *(S is closed under taking squares.)*

- Note: this result holds more generally for $\mathcal{V}$ a special Euclidean Jordan algebra, and $\mathcal{K}$ its cone of squares.
- We view $\mathbb{S}^n$ as a Euclidean Jordan algebra with product $X \circ Y = \frac{1}{2}(XY + YX)$.
- Thus condition 2 means S is a Jordan sub-algebra of $\mathbb{S}^n$, since $X \circ Y = \frac{1}{2}\left((X+Y)^2 - X^2 - Y^2\right)$.

Partition subspaces

Definition

A partition subspace of $\mathbb{S}^n$ has a 0/1 basis that sums to the all-ones matrix.

For example the following spaces are partition spaces:

$$P_1 = \begin{pmatrix} a & a & b \\ a & a & b \\ b & b & c \end{pmatrix}, \quad P_2 = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & c \end{pmatrix}, \quad P_1 \wedge P_2 = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & d \end{pmatrix},$$

where $P_1 \wedge P_2$ is the coarsest partition space *refining* both P_1 and P_2.

Finding the minimum admissible partition subspace

The following algorithm (Permenter, 2017) gives us the optimal admissible partition subspace when $\mathcal{K} = \mathbb{S}^n_+$:

$P \leftarrow \mathrm{part}(P_{\mathcal{L}}(C)) \wedge \mathrm{part}(P_{\mathcal{L}^\perp}(X_0))$
repeat
  $P \leftarrow P \wedge \mathrm{part}(P_{\mathcal{L}}(P))$
  $P \leftarrow P \wedge \mathrm{part}(\mathrm{span}\{X^2 \mid X \in P\})$
until *converged*;

Here part returns the partition given by unique matrix entries, and $\wedge$ the coarsest refinement of two partitions.

Jordan configurations

Definition

If a partition subspace of $\mathbb{S}^n$ is also a Jordan sub-algebra of $\mathbb{S}^n$, then it is called a Jordan configuration.

- The symmetric part of a coherent configuration is a Jordan configuration ...
- ... but it is not known if the converse is true.
- The algorithm on the previous slides actually outputs a Jordan configuration (closed under taking squares) when $\mathcal{K} = \mathbb{S}^n_+$.
- This has to be the case, since any admissible subspace must be a Jordan sub-algebra when $\mathcal{K} = \mathbb{S}^n_+$.

Extensions to $\mathcal{D}^n$

Extensions to $\mathcal{D}^n$

$$\inf\left\{\langle C, X\rangle \ : \ X \in X_0 + \mathcal{L},\ X \in \mathcal{K}\right\}.$$

Theorem (Brosch-De Klerk)

Consider a conic optimization problem with $\mathcal{V} = \mathbb{S}^n$, and $\mathcal{K} = \mathbb{S}^n_+$, and let S be a admissible partition subspace for this problem. Then, S is also an admissible partition subspace for the related problem where we replace $\mathcal{K} = \mathbb{S}^n_+$ by $\mathcal{K} = \mathcal{D}^n$.

Thus we may use the above algorithm to find admissible partition subspaces when $\mathcal{K} = \mathcal{D}^n$.

How restrictive are partition subspaces?

Recall that any admissible subspace S of

$$\inf\{\langle C, X\rangle \ : \ X \in X_0 + \mathcal{L},\ X \in \mathcal{D}^n\}$$

satisfies $P_S(\mathcal{D}^n) \subseteq \mathcal{D}^n$.

Theorem (Brosch-De Klerk)

If subspace S contains I, J, and $P_S(\mathcal{D}^n) \subseteq \mathcal{D}^n$, then:

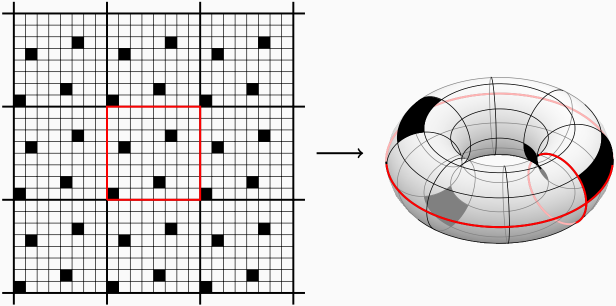
1. S is a partition subspace.
2. If, in addition, $P_S(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+$, then S is a Jordan configuration.

It remains an open question when $P_S(\mathcal{D}^n) \subseteq \mathcal{D}^n$ implies $P_S(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+$.

QAP example: energy minimization on a toric grid

QAP example

Place *m repulsive particles* on a $n_1 \times n_2$ toric grid to minimize the total energy of the system.



Example: $n_1 \times n_2 = 8 \times 8$ toric grid with $m = 4$.

Reformulation as QAP

For the reformulation as QAP we assume an ordering on the $n = n_1 n_2$ grid points.

Define $A = (a_{ij}), B = (b_{ij}) \in \mathbb{S}^n$ (indexed by grid points):

$$a_{ij} = \begin{cases} 1, & \text{if } i, j \leq m \\ 0, & \text{otherwise,} \end{cases},$$

and b_{ij} is the inverse of the Lee distance (shortest path on the toric grid) between grid points i and j. Thus, if particles are placed at grid points i and j, this contributes b_{ij} to the total energy.

QAP reformulation:

$$\min_{\phi \in S_n} \sum_{i,j=1}^{n} a_{ij} b_{\phi(i)\phi(j)}$$

About energy minimization on a toric grid

- Applications from physics is the search for ground states of a two-dimensional repulsive lattice gas at zero temperature.

G.I. Watson. Repulsive particles on a two-dimensional lattice. *Physica A: Statistical Mechanics and its Applications*, 246(1-2):253–274, 1997.

- Application to grey-scale printing [Taillard (1995)]
- For $m = n/2$ and $n_1 = n_2$, chessboard configurations are known to be optimal.

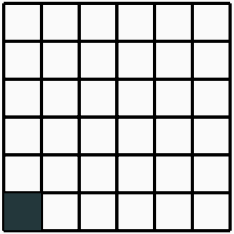
N. Bouman, J. Draisma, and J.S.H. Van Leeuwaarden. Energy minimization of repelling particles on a toric grid. *SIAM Journal on Discrete Mathematics*, 27(3):1295–1312, 2013.

This was shown using an eigenvalue bound called $PB(A, B)$.

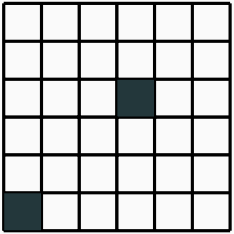
Bounds

$n_1 = n_2$	m	$PB(A,B)$	$SDPQAP(A,B)$	Upper bounds from simulated annealing
6	1	−1.51	0.00	0.00
	2	−2.13	0.33	0.33
	4	−0.64	3.00	3.00
	12	41.47	44.00	44.00
	18	111.00	111.00	111.00
7	1	−1.54	0.00	0.00
	2	−2.29	0.33	0.33
8	1	−1.67	0.00	0.00
	2	−2.65	0.25	0.25
	4	−2.57	2.27	2.27
	32	286.67	286.67	286.67
10	1	−1.72	0.00	0.00
	2	−2.88	0.20	0.20
	4	−3.57	1.81	1.81
	20	70.23	81.43	81.43
	50	588.33	588.33	588.33

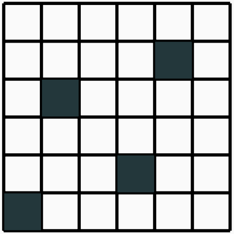
Optimal arrangements on a 6×6 grid



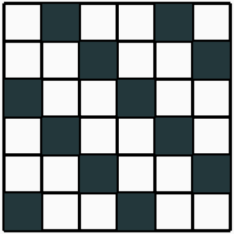
$m = 1$



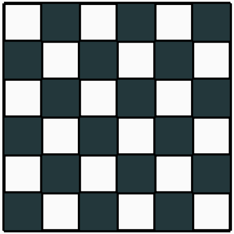
$m = 2$



$m = 4$

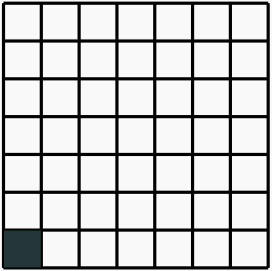


$m = 12$

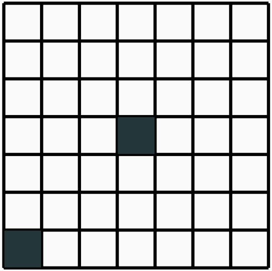


$m = 18$

Optimal arrangements on a 7×7 grid

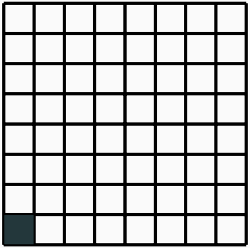


$m = 1$

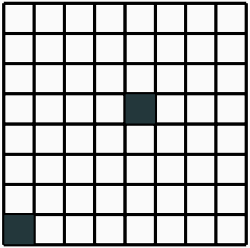


$m = 2$

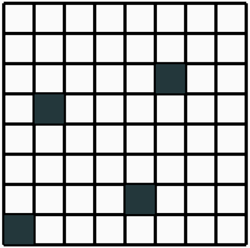
Optimal arrangements on a 8×8 grid



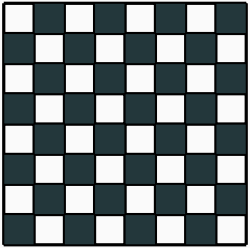
$m = 1$



$m = 2$

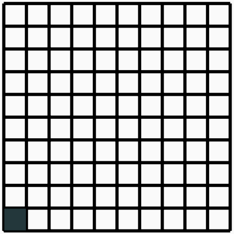


$m = 4$

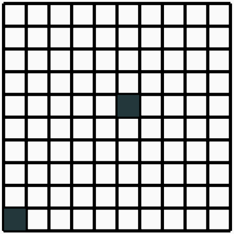


$m = 32$

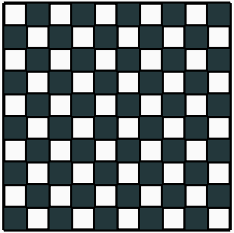
Optimal arrangements on a 10×10 grid



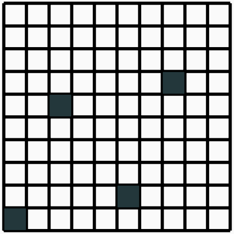
$m = 1$



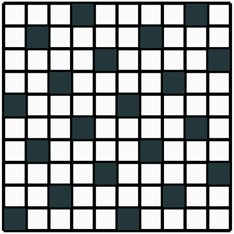
$m = 2$



$m = 50$



$m = 4$



$m = 20$

Software and references

New Julia software package

Julia package *SDPSymmetryReduction* by Daniel Brosch, available at

`github.com/DanielBrosch/SDPSymmetryReduction.jl`

Includes routines to:

- Find an admissible Jordan configuration;
- Block diagonalize the basis of the Jordan configuration;
- Reduce and solve the QAP relaxation (using JuMP and MOSEK to solve the reduced relaxation);
- Compute the Lovász-Schrijver ϑ' function of a graph after performing the Jordan reduction.

References

For this talk:

- Jordan reduction for $\mathcal{D}^n$ & Julia software: arXiv:2001.11348
- Discrete energy minimization on a toric grid: arXiv:1908.00872 (appeared in *Discrete Optimization*).

Background reading on Jordan reduction:

F.N. Permenter. *Reduction methods in semidefinite and conic optimization*. PhD thesis, Massachusetts Institute of Technology, 2017.

The End